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# Nilpotent symplectic alternating algebras I

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We develop a structure theory for nilpotent symplectic alternating algebras.

## 1 Introduction

Symplectic alternating algebras have arisen from the study of 2-Engel groups (see [2],[4]) but seem also to be of interest in their own right, with many beautiful properties. Some general theory was developed in [3] and [5].

**Definition.** Let  $F$  be a field. A *symplectic alternating algebra* over  $F$  is a triple  $(L, ( , ), \cdot)$  where  $L$  is a symplectic vector space over  $F$  with respect to a non-degenerate alternating form  $( , )$  and  $\cdot$  is a bilinear and alternating binary operation on  $L$  such that

$$(u \cdot v, w) = (v \cdot w, u)$$

for all  $u, v, w \in L$ .

Notice that  $(u \cdot x, v) = (x \cdot v, u) = -(v \cdot x, u) = (u, v \cdot x)$  and thus the multiplication from the right by  $x$  is self-adjoint with respect to the alternating form. As the alternating form is non-degenerate,  $L$  is of even dimension and we can pick a basis  $x_1, y_1, \dots, x_n, y_n$  with the property that  $(x_i, x_j) = (y_i, y_j) = 0$

and  $(x_i, y_j) = \delta_{ij}$  for  $1 \leq i \leq j \leq n$ . We refer to a basis of this type as a *standard basis*.

Suppose we have any basis  $u_1, \dots, u_{2n}$  for  $L$ . The structure of  $L$  is then determined from

$$(u_i u_j, u_k) = \gamma_{ijk}, \quad 1 \leq i < j < k \leq 2n.$$

The map  $L^3 \rightarrow F$ ,  $(u, v, w) \mapsto (u \cdot v, w)$  is an alternating ternary form and each alternating ternary form on a given symplectic vector space, with a non-degenerate alternating form, defines a unique symplectic alternating algebra. Classifying symplectic alternating algebras of dimension  $2n$  over a field  $F$  is then equivalent to finding all the  $\mathrm{Sp}(V)$ -orbits of  $(\wedge^3 V)^*$  under the natural action, where  $V$  is a symplectic vector space of dimension  $2n$  with a non-degenerate alternating form. Suppose that  $F$  is a finite field and suppose that the disjoint  $\mathrm{Sp}(V)$ -orbits of  $(\wedge^3 V)^*$  are  $u_1^{\mathrm{Sp}(V)}, \dots, u_m^{\mathrm{Sp}(V)}$ . Then

$$m \leq |F|^{\binom{2n}{3}} = |(\wedge^3 V)^*| \leq m |\mathrm{Sp}(V)| \leq m |F|^{\binom{2n+1}{2}}.$$

It follows that  $m = |F|^{\frac{4n^3}{3} + O(n^2)}$ . Because of the sheer growth, a general classification of symplectic alternating algebras seems impossible. There is a close connection between symplectic alternating algebras over the field  $\mathrm{GF}(3)$  of three elements and a certain class of 2-Engel groups and in [5] the symplectic alternating algebras over  $\mathrm{GF}(3)$  of dimension up to 6 were classified. There are 31 such algebras of dimension 6 of which 15 are simple. We would like to mention here also the work of Atkinson [1] who in his thesis looked at alternating ternary forms over  $\mathrm{GF}(3)$  in order to study a certain class of groups of exponent 3.

As we said before, some general theory was developed in [3] and [5]. In particular a well-known dichotomy property for Lie algebras also holds for symplectic alternating algebras. Thus a symplectic alternating algebra is either semi-simple or has a non-trivial abelian ideal. Another interesting property is that any symplectic alternating algebra that is nilpotent-by-abelian must be nilpotent.

In this paper and its sequel we study nilpotent symplectic alternating al-

gebras. This paper deals with the structure theory and also gives the classification of nilpotent symplectic alternating algebras of dimension up to 8 over any field. The sequel will be mostly about extending the classification of algebras to dimension 10 that is far more involved than the algebras of lower dimension. Turning back to this paper, we will in Section 2 describe some general results that in particular lead to specific type of presentations that we call nilpotent presentations. All algebras with a nilpotent presentation are nilpotent and conversely any nilpotent algebra will have a nilpotent presentation. In Section 3 we will focus on the algebras that are of maximal class and we will see that their structure is very rigid. Finally to illustrate the theory we will in Section 4 classify the nilpotent symplectic alternating algebras of dimension up to 8 over an arbitrary field  $F$ .

In this paper we will adopt the *left-normed* convention for products. Thus  $u_1u_2\ldots u_n$  stands for  $(\ldots(u_1u_2)\cdots)u_n$ . Also  $U \leq V$  stands for ‘ $U$  is a subspace of  $V$ ’.

Many of the terms that we use in this paper are analogous to the corresponding terms for related structures. Thus a subspace  $I$  of a symplectic alternating algebra  $L$  is an ideal if  $IL \leq I$ . From [5] we know that  $I^\perp$  is an ideal whenever  $I$  is an ideal. The definition of a nilpotent symplectic alternating algebra causes no problem either.

**Definition** A symplectic alternating algebra  $L$  is *nilpotent* if there exists an ascending chain of ideals  $I_0, \ldots, I_n$  such that

$$\{0\} = I_0 \leq I_1 \leq \cdots \leq I_n = L$$

and  $I_sL \leq I_{s-1}$  for  $s = 1, \ldots, n$ . The smallest possible  $n$  is then called the *nilpotence class* of  $L$ .

**Definition.** More generally, if  $I_0 \leq I_1 \leq \cdots \leq I_n$  is any chain of ideals of  $L$  then we say that this chain is central in  $L$  if  $I_sL \leq I_{s-1}$  for  $s = 1, \ldots, n$ .

We define the lower and upper central series in an analogous way to related structures like associative algebras and Lie algebras. Thus we define the lower central series recursively by  $L^1 = L$  and  $L^{n+1} = L^nL$ , and the upper central series by  $Z_0(L) = \{0\}$  and  $Z_{n+1}(L) = \{x \in L : xL \subseteq Z_n(L)\}$ .

It is readily seen that the terms of the lower and the upper central series are all ideals of  $L$ . The following beautiful property was proved in [5] and will be used frequently

$$Z_n(L) = (L^{n+1})^\perp. \quad (1)$$

**Remark.** Notice however that the lack of the Jacobi identity means that many properties that hold for Lie algebras do not hold for symplectic alternating algebras. As the following example shows, it is not true in general that the product of two ideals is an ideal. That example also shows that the formula  $L^i L^j \leq L^{i+j}$  does not hold in general.

**Example.** Consider the 12-dimensional symplectic alternating algebra which has a standard basis  $x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4, x_5, y_5, x_6, y_6$  where

$$(x_3 y_5, y_6) = (x_2 y_4, y_6) = (x_1 y_4, y_5) = (y_1 y_2, y_3) = 1$$

and  $(uv, w) = 0$  if  $u, v, w$  are basis elements where  $\{u, v, w\} \notin \{\{x_3, y_5, y_6\}, \{x_2, y_4, y_6\}, \{x_1, y_4, y_5\}, \{y_1, y_2, y_3\}\}$ . Notice that this implies that

$$\begin{aligned} x_3 y_5 &= x_6, & x_1 y_4 &= x_5, & y_2 y_3 &= x_1, \\ x_3 y_6 &= -x_5, & x_1 y_5 &= -x_4, & y_4 y_5 &= -y_1, \\ x_2 y_4 &= x_6, & y_1 y_2 &= x_3, & y_4 y_6 &= -y_2, \\ x_2 y_6 &= -x_4, & y_1 y_3 &= -x_2, & y_5 y_6 &= -y_3. \end{aligned}$$

From this one sees that

$$\begin{aligned} L^2 &= Fx_6 + Fx_5 + \cdots + Fx_1 + Fy_1 + Fy_2 + Fy_3, \\ L^3 &= Fx_6 + Fx_5 + \cdots + Fx_1, \\ L^4 &= Fx_6 + Fx_5 + Fx_4, \\ L^5 &= 0, \\ L^2 L^2 &= Fx_3 + Fx_2 + Fx_1. \end{aligned}$$

In particular  $L$  is nilpotent of class 4,  $L^2 L^2$  is not an ideal and  $L^2 L^2 \not\leq L^4$ .

This example indicates that symplectic alternating algebras do differ from Lie algebras. We are going to see in the following sections that there are some shared properties but the next lemma underlines the difference by showing

that the two classes of algebras do not have many algebras in common when the characteristic is not 2. In fact only the symplectic alternating algebras that are obviously Lie algebras are there, namely those of class at most 2.

**Lemma 1.1** *Let  $L$  be a symplectic alternating algebra where  $\text{char } L \neq 2$  and  $L$  is either associative or a Lie algebra. Then  $L^3 = \{0\}$ .*

**Proof.** Let us first assume that  $L$  is associative. We then have

$$0 = (xyz - x(yz), t) = (x, tzy - t(yz)) = (x, tzy - tyz)$$

for all  $x, y, z, t \in L$ . It follows that  $tzy = tyz = -ytz$  for all  $t, z, y \in L$ . Using this last property repeatedly we get that

$$xyz = -zxy = yzx = -xyz$$

and thus  $2xyz = 0$  for all  $x, y, z \in L$ . As  $\text{char } L \neq 2$ , it follows that  $L^3 = 0$ .

Now suppose  $L$  is a Lie algebra. We then have

$$0 = (xyz + yzx + zxy, t) = (x, tzy - t(yz) - tyz) = 2(x, tzy - tyz).$$

As  $\text{char } L \neq 2$ , it follows again that  $tzy = tyz$  for all  $t, z, y \in L$  and this implies again that  $L^3 = \{0\}$ .  $\square$ .

One handicap that the symplectic alternating algebras have is that when  $I$  is an ideal then  $L/I$  is in general only an alternating algebra as there is no natural way of inducing an alternating form on this quotient. For example simply for the reason that the quotient can have odd dimension. There is however a weaker form of a quotient structure that we can associate to any ideal  $I$  of  $L$  that works. Thus for any ideal  $I$  we have that  $(I^\perp + I)/I$  is a well defined symplectic alternating algebra with the natural induced multiplication and where the induced alternating form is given by  $(u + I, v + I) = (u, v)$  for  $u, v \in I^\perp$ . The reader can easily convince himself that this is well defined and that  $((I^\perp + I)/I)^\perp = 0$ . This algebra is also isomorphic to  $I^\perp/(I \cap I^\perp)$  that has a similar naturally induced structure as a symplectic alternating algebra.

**Remark.** There are some familiar facts for Lie algebras that do not rely on the Jacobi identity and remain true for symplectic alternating algebras.

Such properties are particularly useful as we can use them when dealing with quotients  $L/I$  where we only know that the resulting algebra is alternating. For example  $L^2$  has co-dimension at least 2 in any nilpotent alternating algebra  $L$  of dimension greater than or equal to 2. From this and the duality given in (1), it follows immediately that the dimension of  $Z(L)$  is at least 2 for any non-trivial nilpotent symplectic algebra which is something that we will also see later as a corollary of Lemma 2.1.

## 2 General Structure Theory

We next see that, like for Lie algebras, all minimal sets of generators have the same number of elements and we can thus introduce the notion of a rank.

**Definition.** Let  $L$  be a nilpotent symplectic alternating algebra. We say that  $\{x_1, \dots, x_r\}$  is a *minimal set of generators* if these generate  $L$  (as an algebra) and no proper subset generates  $L$ .

**Lemma 2.1** *Let  $L$  be a nilpotent symplectic alternating algebra. Any minimal set of generators has the same size which is  $\dim L - \dim L^2$ .*

**Proof** Let  $x_1, \dots, x_r \in L$  and let  $M$  be the subalgebra of  $L$  generated by these elements. It suffices to show that  $L = M$  if and only if  $x_1 + L^2, \dots, x_r + L^2$  generate  $L/L^2$  as a vector space. Suppose first that  $L = M$ . Notice that  $M = Fx_1 + \dots + Fx_r + M \cap L^2$  and thus it is clear that  $L/L^2$  is generated by  $x_1 + L^2, \dots, x_r + L^2$  as a vector space. Conversely suppose now that the images of  $x_1, \dots, x_r$  in  $L/L^2$  generate  $L/L^2$  as a vector space. An easy induction shows that

$$L = M + L^{s+1}, \quad L^s = M^s + L^{s+1}$$

for all integers  $s \geq 1$ . If the class of  $L$  is  $n$ , we get in particular that  $L = M + L^{n+1} = M$ .  $\square$

**Definition.** Let  $L$  be a nilpotent symplectic alternating algebra. The unique smallest number of generators for  $L$ , as an algebra, is called the *rank* of  $L$  and is denoted  $r(L)$ .

By last lemma we know that  $r(L) = \dim L - \dim L^2$ . This has the following curious consequence.

**Corollary 2.2** *Let  $L$  be a nilpotent symplectic alternating algebra. We have  $r(L) = \dim Z(L)$ . In particular if  $L \neq \{0\}$  then  $\dim Z(L) \geq 2$ .*

**Proof.** From (1) we know that  $Z(L) = (L^2)^\perp$ . Therefore

$$r(L) = \dim L - \dim L^2 = \dim (L^2)^\perp = \dim Z(L).$$

Finally, we cannot have  $r(L) = 1$  as then we would have that  $L$  is one-dimensional. Hence  $\dim Z(L) \geq 2$ .  $\square$ .

**Lemma 2.3** *Let  $I, J$  be ideals of a nilpotent symplectic alternating algebra where  $I \subseteq J$ . If  $\dim J = \dim I + 1$  then  $I \leq J$  is central. If  $I$  is an ideal such that  $\dim I < 2n = \dim L$  then there exists an ideal  $J$  such that  $\dim J = \dim I + 1$ . If furthermore  $I$  is an isotropic ideal and  $\dim I < n$  then  $J$  can be chosen to be isotropic.*

**Proof** Suppose  $J = I + Fx$  for some  $x \in L$ . Let  $y \in L$ . To show that  $I \leq J$  is central, it suffices to show that  $x \cdot y \in I$ . Suppose that  $xy = u_1 + ax$  for  $u_1 \in I$  and  $a \in F$ . As  $I \trianglelefteq L$  it follows by induction that  $xy^r = u_r + a^r x$  for some  $u_r \in I$ . If  $L$  is nilpotent of class at most  $m$  it follows that  $0 = u_m + a^m x$  and hence  $a = 0$ .

For the latter part suppose first that  $I$  is any ideal such that  $\dim I < 2n$ . Let  $m$  be the largest positive integer such that  $L^m \not\subseteq I$ . Pick  $u \in L^m \setminus I$ . Then  $J = I + Fu$  is the required ideal such that  $I \leq J$  is central. Now suppose furthermore that  $I$  is isotropic and that  $\dim I < n$ . Then  $I^\perp$  is also an ideal and  $I < I^\perp$ . Let  $m$  be the largest non-negative integer such that  $I^\perp \underbrace{L \cdots L}_m \not\subseteq I$ . Let  $u \in I^\perp \underbrace{L \cdots L}_m \setminus I$  and again the ideal  $J = I + Fu$  is the one required.  $\square$

**Remark.** Let  $U, V$  be subspaces of  $L$ . Notice that

$$UV = 0 \Leftrightarrow (UV, L) = 0 \Leftrightarrow (UL, V) = 0 \Leftrightarrow UL \leq V^\perp.$$

In other words we have that  $U$  annihilates  $V$  if and only if it annihilates  $L/V^\perp$ . This is a useful property that we will be making use of later.



**Lemma 2.4** *Let  $I$  and  $J$  be ideals in a symplectic alternating algebra  $L$ . We have that  $I \cdot L \leq J$  if and only if  $I \cdot J^\perp = \{0\}$ . In particular  $L^m \cdot Z_m(L) = \{0\}$  for all  $m \geq 1$ .*

**Proof** From the property given in last remark, we know that  $I$  annihilates  $L/J$  if and only if  $I$  annihilates  $J^\perp$ . The second part follows from this, the fact that  $L^m$  annihilates  $L/L^{m+1}$ , and the fact that  $(L^{m+1})^\perp = Z_m(L)$ .  $\square$

**Remark.** It follows in particular that  $I \cdot I^\perp = \{0\}$  for any ideal  $I$ . In particular any isotropic ideal is abelian. Notice also that the property  $L^m Z_m(L) = \{0\}$  is equivalent to the fact that  $Z_m(L)$  annihilates  $L/Z_{m-1}(L)$ .

**Remark.** We have seen in the introduction that it is not true in general that  $L^i L^j \leq L^{i+j}$ . As  $(L^m)^\perp = Z_{m-1}(L)$  for all  $m \geq 1$ , we however have that

$$\begin{aligned} L^i L^j \leq L^{i+j} &\Leftrightarrow (L^i L^j, Z_{i+j-1}(L)) = 0 \Leftrightarrow (L^i Z_{i+j-1}(L), L^j) = 0 \\ &\Leftrightarrow L^i Z_{i+j-1}(L) \leq Z_{j-1}(L). \end{aligned}$$

The obvious fact that  $L^m L \leq L^{m+1}$  thus gives us the interesting fact from last lemma that  $L^m Z_m(L) = \{0\}$ .

**Lemma 2.5** *Let  $I$  be an ideal of  $L$ . Then  $IL \leq I^\perp$  if and only if  $I$  is abelian.*

**Proof** We have that  $I$  annihilates  $I$  if and only if  $I$  annihilates  $L/I^\perp$ .  $\square$

**Remark.** As  $I$  is an ideal we have in fact that  $IL \leq I^\perp$  if and only if  $IL \leq I \cap I^\perp$ . Here  $I \cap I^\perp$  is the ‘isotropic part’ of  $I$ .

**Lemma 2.6** *Let  $L$  be a nilpotent symplectic alternating algebra with ideals  $I, J$  where  $J = I + Fx + Fy$ ,  $(x, y) = 1$  and  $Fx + Fy \leq I^\perp$ . Then  $JL \leq I$ . Furthermore if  $I$  is isotropic then  $J$  is abelian.*

**Proof** As  $J$  is an ideal of  $L$  and as  $(xt, x) = 0$  for all  $t \in L$  we have that  $I + Fx$  is an ideal of  $L$ . By Lemma 2.3 we have that  $I \leq I + Fx$  is central. Similarly  $I \leq I + Fy$  is central and thus  $JL \leq I$ . For the second part notice that if  $I$  is isotropic then  $I = J \cap J^\perp$  thus  $JL \leq I = J \cap J^\perp$  and by Lemma 2.5 it follows that  $J$  is abelian.  $\square$

**Lemma 2.7** *Let  $L$  be a nilpotent symplectic alternating algebra. Every ideal  $I$  of dimension 2 is contained in  $Z(L)$ . Equivalently, every ideal of co-dimension 2 must contain  $L^2$ .*

**Proof.** The second statement is a trivial fact that holds in all nilpotent alternating algebras. The first statement is a consequence of this and of the duality given by  $I \leq Z(L) \Leftrightarrow L^2 = Z(L)^\perp \leq I^\perp$ .  $\square$

**Lemma 2.8** *Let  $I, J$  be ideals of a symplectic alternating algebra  $L$  and let  $x \in L$ . We have  $Jx \leq I$  if and only if  $I^\perp x \leq J^\perp$ .*

**Proof.** We have that  $Jx \leq I$  is equivalent to  $(ux, v) = 0$  for all  $u \in J$  and all  $v \in I^\perp$ . But this is equivalent to saying that  $(vx, u) = 0$  for all  $v \in I^\perp$  and  $u \in J = (J^\perp)^\perp$  and this is the same as saying that  $I^\perp x \leq J^\perp$ .  $\square$

**Proposition 2.9** *Let  $L$  be a symplectic alternating algebra. No term of the upper central series has co-dimension 1. Equivalently, no term of the lower central series has dimension 1.*

**Proof** The first fact is a well-known fact about alternating algebras and follows from the fact that if  $A$  is an alternating algebra then  $A/Z(A)$  cannot be one-dimensional. Now the interesting second statement is a consequence of this and the duality  $(L^r)^\perp = Z_{r-1}(L)$ .  $\square$

In particular we have that

$$\{0\} = I_0 \leq I_1 \leq \cdots \leq I_m = L$$

is an ascending central chain if and only if

$$L = I_0^\perp \geq I_1^\perp \geq \cdots \geq I_m^\perp = \{0\}$$

is a descending central chain.

**Remark.** Suppose that  $L$  is any nilpotent alternating algebra such that  $L/L^2$  is 2-dimensional. Then it follows immediately that the dimension of  $L^2/L^3$  is at most 1 and that the dimension of  $L^3/L^4$  is at most 2. Using this general fact and Proposition 2.9 one can quickly show that all nilpotent symplectic alternating algebras of dimension up to 4 must be abelian. This

is clear when the dimension is 2. Now suppose that  $L$  is a nilpotent symplectic alternating algebra of dimension 4. We know that  $\dim L/L^2 \geq 2$ . If  $\dim L^2 = 2$  then by the reasoning above, we would have that  $\dim L^3 = 1$  that contradicts Proposition 2.9. By that proposition we neither can have that  $\dim L^2 = 1$ . Thus we must have  $L^2 = 0$  and  $L$  is abelian.

**Theorem 2.10** *Let  $L$  be a nilpotent symplectic alternating algebra of dimension  $2n \geq 2$ . There exists an ascending chain of isotropic ideals*

$$\{0\} = I_0 < I_1 < \cdots < I_{n-1} < I_n$$

*such that  $\dim I_r = r$  for  $r = 0, \dots, n$ . Furthermore, for  $2n \geq 6$ ,  $I_{n-1}^\perp$  is abelian and the ascending chain*

$$\{0\} < I_2 < I_3 < \cdots < I_{n-1} < I_{n-1}^\perp < I_{n-2}^\perp < \cdots < I_2^\perp < L$$

*is a central chain. In particular  $L$  is nilpotent of class at most  $2n - 3$ .*

**Proof** Starting with the ideal  $I_0 = \{0\}$ , we can apply Lemma 2.3 iteratively to get the required chain

$$\{0\} = I_0 < I_1 < \cdots < I_n.$$

By Lemma 2.7 we have that  $I_2 \leq Z(L)$ . By this and Lemma 2.3 we thus have that the chain

$$I_0 < I_2 < I_3 < \cdots < I_{n-1}$$

is central in  $L$ . By Lemma 2.8 it follows that the chain

$$I_{n-1}^\perp < I_{n-2}^\perp < \cdots < I_2^\perp < I_0^\perp$$

is also central. It only remains to see that  $I_{n-1} < I_{n-1}^\perp$  is central and that  $I_{n-1}^\perp$  is abelian. As  $I_{n-1}^\perp = I_{n-1} + Fx + Fy$  for some  $x, y \in L$  where  $(x, y) = 1$  and as  $I_{n-1}$  is isotropic, this follows from Lemma 2.6.  $\square$

**Remark.** When  $\dim Z(L) = r < n$ , we can choose our chain such that  $I_r = Z(L)$ . We then get a central chain

$$I_0 < I_r < I_{r+1} < \cdots < I_{n-1} < I_{n-1}^\perp < I_{n-2}^\perp < \cdots < I_r^\perp < L.$$

In particular the class is then at most  $2n - 3 - 2(r - 2) = 2n - 2r + 1$ .

**Presentations of nilpotent symplectic alternating algebras.** Last proposition tells us a great deal about the structure of nilpotent symplectic alternating algebras. A moments reflection should convince the reader that we can pick a standard basis  $x_1, y_1, x_2, y_2, \dots, x_n, y_n$  such that

$$I_1 = Fx_n, I_2 = Fx_n + Fx_{n-1}, \dots, I_n = Fx_n + \dots + Fx_1,$$

$$I_{n-1}^\perp = I_n + Fy_1, I_{n-2}^\perp = I_n + Fy_1 + Fy_2, \dots, I_0^\perp = L = I_n + Fy_1 + \dots + Fy_n.$$

Now let  $u, v, w$  be three of the basis elements. Since  $I_n$  is abelian we have that  $(uv, w) = 0$  whenever two of these three elements are from  $\{x_1, \dots, x_n\}$ . The fact that

$$\{0\} < I_1 < \dots < I_n$$

is central also implies that  $(x_i y_j, y_k) = 0$  if  $i \geq k$ . So we only need to consider the possible non-zero triples  $(x_i y_j, y_k), (y_i y_j, y_k)$  for  $1 \leq i < j < k \leq n$ . For each triple  $(i, j, k)$  with  $1 \leq i < j < k \leq n$ , let  $\alpha(i, j, k)$  and  $\beta(i, j, k)$  be some elements in the field  $F$ . We refer to the data

$$\mathcal{P} : (x_i y_j, y_k) = \alpha(i, j, k), (y_i y_j, y_k) = \beta(i, j, k), \quad 1 \leq i < j < k \leq n$$

as a *nilpotent presentation*. We have just seen that every nilpotent symplectic alternating algebra has a presentation of this type. Conversely, given any nilpotent presentation, let

$$I_r = Fx_n + Fx_{n-1} + \dots + Fx_{n+1-r}$$

and we get an ascending central chain of isotropic ideals  $\{0\} = I_0 < I_1 < \dots < I_n$  such that  $\dim I_j = j$  for  $j = 1, \dots, n$ . By Lemma 2.8 we then get a central chain

$$\{0\} = I_0 < I_1 < \dots < I_n < I_{n-1}^\perp < I_{n-2}^\perp < \dots < I_0^\perp = L$$

and thus  $L$  is nilpotent. Thus every nilpotent presentation describes a nilpotent symplectic alternating algebra.

**Remark.** Notice that there are  $2\binom{n}{3}$  parameters for these presentations.

If  $F$  is a finite field this thus gives the value  $|F|^{2\binom{n}{3}}$  as the upper bound for the number of  $2n$ -dimensional nilpotent symplectic alternating algebras over the field  $F$ . Armed with this information it is not difficult to get some good information about the growth of nilpotent symplectic alternating algebras over a finite field  $F$ . Let  $V$  be a  $2n$ -dimensional vector space over  $F$  and consider  $(\wedge^3 V)^*$ . After fixing a standard basis for  $V$ , each presentation of a symplectic alternating algebra corresponds to an element in  $(\wedge^3 V)^*$ . Now let  $\mathcal{N}$  be the subset of  $(\wedge^3 V)^*$  corresponding to all nilpotent presentations. The number of nilpotent symplectic alternating algebras of dimension  $2n$  is the same as the number of  $Sp(V)$ -orbits of  $(\wedge^3 V)^*$  consisting of presentations that give nilpotent algebras. Suppose these are  $u_i^{\text{Sp}(V)}$ ,  $i = 1, \dots, m$ . Then

$$\mathcal{N} \subseteq \cup_{i=1}^m u_i^{\text{Sp}(V)}$$

and thus  $|F|^{2\binom{n}{3}} = |\mathcal{N}| \leq m \cdot |\text{Sp}(V)| \leq m \cdot |F|^{\binom{2n+1}{2}}$ . These calculations show that the number of nilpotent symplectic alternating algebras is

$$m = |F|^{n^3/3 + O(n^2)}.$$

From Proposition 2.9 we know that no term of the lower central series of a symplectic alternating algebra can be 1-dimensional. Next proposition shows that some of terms of the lower central series cannot be 2-dimensional.

**Proposition 2.11** *Let  $L$  be a symplectic alternating algebra we have that  $\dim L^m \neq 2$  for  $2 \leq m \leq 4$ . Equivalently  $Z_m(L)$  is not of co-dimension 2 if  $1 \leq m \leq 3$ .*

**Proof.** We first prove that  $\dim L^2 \neq 2$ . We argue by contradiction and suppose  $\dim L^2 = 2$ . Then

$$2 = \dim L^2 = \dim Z(L)^\perp = \dim L - \dim Z(L).$$

Suppose  $L = Z(L) + Fu + Fv$ . Then  $L^2 = Fuv$ , which contradicts  $\dim L^2 = 2$ .

Next we turn to showing that  $\dim L^3 \neq 2$ . We argue by contradiction and let  $L$  be a counter example of smallest dimension. We first notice that  $Z(L)$  must be isotropic as otherwise  $L = I \oplus I^\perp$  for some 2-dimensional ideal  $I = Fu + Fv \leq Z(L)$  where  $(u, v) = 1$ . But then  $M = I^\perp$  is a symplectic

alternating algebra of smaller dimension where  $M^3 = L^3$  is of dimension 2. This however contradicts the minimality of  $L$ . We can thus assume that  $Z(L)$  is isotropic. Notice that

$$2 = \dim L^3 = \dim Z_2(L)^\perp = \dim L - \dim Z_2(L).$$

Say,  $L = Z_2(L) + Fx + Fy$ . Then  $L^2 = Z(L) + Fxy$  and, as  $Z(L)$  is isotropic and  $xy \in L^2 = Z(L)^\perp$ ,  $L^2$  is isotropic. Thus  $L^2 \leq (L^2)^\perp = Z(L)$  and we get the contradiction that  $L^3 = \{0\}$ .

It now only remains to deal with  $L^4$ . For a contradiction, suppose that  $\dim L^4 = 2$ . Then

$$2 = \dim L^4 = \dim Z_3(L)^\perp = \dim L - \dim Z_3(L).$$

Say  $L = Z_3(L) + Fu + Fv$ . Then  $L^2 \leq Z_2(L) + Fuv$  and using the fact that  $Z_2(L) \cdot L^2 = \{0\}$  we get

$$L^2 \cdot L^2 \leq (Z_2(L) + Fuv)L^2 = Fuv \cdot L^2 \leq Fuv \cdot (Z_2(L) + Fuv) = F(uv)(uv) = 0.$$

Thus  $0 = (L, L^2 \cdot L^2) \Rightarrow (L^3, L^2) = 0 \Rightarrow (L^4, L) = 0$ , that gives us the contradiction that  $L^4 = \{0\}$ .  $\square$

**Example.** Let  $L$  be the nilpotent alternating algebra with presentation (we only list the triples that have non-zero value)

$$(x_2y_3, y_4) = 1, (x_1y_2, y_3) = 1, (y_1y_2, y_4) = 1.$$

Then inspection shows that  $\dim L^5 = 2$ . The bound 4 in last proposition is therefore the best one.

### 3 The structure of nilpotent symplectic alternating algebras of maximal class

We have seen previously that nilpotent symplectic alternating algebras of dimension  $2n$  have class at most  $2n - 3$ . For every algebra of dimension  $2n \geq 8$  this bound is attained. As well as demonstrating this we will see that the

structure of these algebras of maximal class is very restricted.

Let  $L$  be a nilpotent symplectic alternating algebra of dimension  $2n \geq 8$  with an ascending chain of isotropic ideals

$$\{0\} = I_0 < I_1 < \cdots < I_n,$$

where  $\dim I_j = j$  for  $j = 1, \dots, n$ .

**Theorem 3.1** *Suppose  $L$  is of maximal class. Then*

$$I_2 = Z_1(L), \quad I_3 = Z_2(L), \quad \dots, \quad I_{n-1} = Z_{n-2}(L),$$

$$I_{n-1}^\perp = Z_{n-1}(L), \quad I_{n-2}^\perp = Z_n(L), \quad \dots, \quad I_2^\perp = Z_{2n-4}(L).$$

Furthermore  $Z_0(L), Z_1(L), \dots, Z_{2n-3}(L)$  are the unique ideals of  $L$  of dimensions  $0, 2, 3, \dots, n-1, n+1, n+2, \dots, 2n-2, 2n$ .

**Proof** Let  $J_0 = \{0\}, J_1 = I_2, \dots, J_{n-2} = I_{n-1}, J_{n-1} = I_{n-1}^\perp, J_n = I_{n-2}^\perp, \dots, J_{2n-4} = I_2^\perp, J_{2n-3} = L$ . By Theorem 2.10, the chain  $J_0 < J_1 < \dots < J_{2n-3}$  is central. We argue by contradiction and let  $i$  be the smallest integer between 1 and  $2n-4$  where  $J_i < Z_i(L)$ . Let  $u \in Z_i(L) \setminus J_i$  and let  $k$  be the smallest integer between  $i$  and  $2n-4$  such that  $u \in J_{k+1}$ . Then

$$J_k < J_k + Fu \leq J_{k+1}.$$

If  $J_{k+1}/J_k$  has dimension 1 it follows that  $J_{k+1} \leq Z_k(L)$  and we get the contradiction that the class is at most  $2n-4$ . We can thus suppose that  $J_{k+1}/J_k$  has dimension 2 and there are two cases to consider, either  $k = n-2$  or  $k = 2n-4$ . In the former case we have

$$I_{n-1} < I_{n-1} + Fu \leq I_{n-1}^\perp$$

which implies that  $I = I_{n-1} + Fu$  is an isotropic ideal of maximal dimension  $n$ . As  $u \in Z_{n-2}(L)$ , we have that  $I_{n-2} < I$  is centralised by  $L$ . By Lemma 2.8 it follows that  $I < I_{n-2}^\perp$  is also centralised by  $L$  and we get a central series

$$\{0\} = I_0 < I_2 < I_3 < \dots < I_{n-2} < I < I_{n-2}^\perp < I_{n-3}^\perp < \dots < I_2^\perp < I_0^\perp = L$$

of length  $2n - 4$  and we get again the contradiction that the class is less than  $2n - 3$ . Finally suppose that  $k = 2n - 4$ . So we have

$$I_2^\perp < I_2^\perp + Fu < L$$

and  $u \in Z_{2n-4}(L)$ . Now let  $v \in L \setminus (I_2^\perp + Fu)$ . Then  $L = I_2^\perp + Fu + Fv$  and  $L^2 = (I_2^\perp + Fu)L \leq Z_{2n-5}(L)$ . Hence  $L \leq Z_{2n-4}(L)$  that again contradicts the assumption that  $L$  is of class  $2n - 3$ .

We now want to show that these terms of the upper central series are the unique ideals of dimensions  $0, 2, 3, \dots, n-1, n+1, n+2, \dots, 2n-2, 2n$ . First let  $I$  be an ideal of dimension 2. By Lemma 2.7 we have that  $I \leq Z(L)$  and as we have seen that  $Z(L)$  has dimension 2, it follows that  $I = Z(L)$ . Now suppose that for some  $2 \leq k \leq n-2$  we know that  $Z_{k-1}(L)$  is the only ideal of dimension  $k$ . Let  $I$  be an ideal of dimension  $k+1$ . As  $L$  is nilpotent we have that  $I$  contains a ideal  $J$  of dimension  $k$ . By the induction hypothesis we have that  $J = Z_{k-1}(L)$  and as  $I/J$  is of dimension 1 we have that  $I \leq Z_k(L)$ . We have that  $Z_k(L)$  has dimension  $k+1$  and thus  $I = Z_k(L)$ . We have thus seen that there are unique ideals of dimensions  $0, 2, 3, \dots, n-1$ . Now let  $I$  be an ideal of dimension  $i \in \{n+1, n+2, \dots, 2n-2, 2n\}$ . Then  $I^\perp$  is an ideal whose dimension is in  $\{0, 2, 3, \dots, n-1\}$ . By what we have just seen  $I^\perp$  is unique and thus  $I$  as well.  $\square$

**Remarks** (1) In particular it follows that  $Z_k(L)^\perp = Z_{2n-3-k}(L)$  for  $0 \leq k \leq 2n - 3$ .

(2) As  $L^k = Z_{k-1}(L)^\perp$ , it follows that  $L, L^2, \dots, L^{2n-2}$  are the unique ideals of dimensions  $2n, 2n-2, 2n-3, \dots, n+1, n-1, n-2, \dots, 2, 0$ . Also

$$L^k = Z_{k-1}(L)^\perp = Z_{2n-k-2}(L).$$

**Remark.** Let  $L$  be any nilpotent symplectic alternating algebra of dimension  $2n \geq 6$  with the property that  $\dim Z(L) = 2$ . Notice that  $Z(L)$  must be isotropic since otherwise we would have a 2-dimensional symplectic subalgebra  $I$  within  $Z(L)$  and we would get a direct sum  $I \oplus I^\perp$  of two symplectic alternating algebras. As  $I^\perp$  has non-trivial center this would contradict the assumption that  $Z(L)$  is 2-dimensional. Now  $L$  has rank 2. Suppose it is generated by  $x, y$ . Then  $L$  is generated by  $x, y, xy$  modulo  $L^3$  and thus



$$\dim Z_2(L) = \dim (L^3)^\perp = \dim L - \dim L^3 = 3.$$

**The complete list of ideals of  $L$ .** We have seen that there is a unique ideal of dimension  $k$  for any  $0 \leq k \leq 2n$  apart from  $k = 1, k = n$  and  $k = 2n - 1$ . Let us now turn to the remaining dimensions. Now every ideal of dimension 1 is contained in  $Z(L)$  and conversely every subspace of dimension 1 in  $Z(L)$  is an ideal.

Next consider an ideal  $I$  of dimension  $2n - 1$ . Then  $I^\perp$  is an ideal of dimension 1 and is thus any subspace of dimension 1 such that

$$\{0\} < I^\perp < Z(L)$$

Equivalently,  $I$  is any subspace of dimension  $2n - 1$  such that

$$L^2 = Z(L)^\perp < I < \{0\}^\perp = L.$$

Finally consider an ideal  $I$  of dimension  $n$ . Since  $L$  is nilpotent there exists an ideal  $J$  of dimension  $n + 1$  containing  $I$ . By last theorem we have that  $J = L^{n-1} = Z_{n-2}(L)^\perp$ . Also  $I$  contains an ideal of dimension  $n - 1$  that we know is  $Z_{n-2}(L)$ . Thus

$$Z_{n-2}(L) < I < Z_{n-2}(L)^\perp.$$

We also know from our previous work that  $Z_{n-2}(L)$  is an isotropic ideal of dimension  $n - 1$ .  $I$  is thus an isotropic ideal of the form

$$Z_{n-2}(L) + Fu$$

For some  $u \in Z_{n-2}(L)^\perp \setminus Z_{n-2}(L)$ . Conversely, as  $Z_{n-2}(L)^\perp L \leq Z_{n-2}(L)$  we have that for any intermediate subspace  $I$  of dimension  $n$  between  $Z_{n-2}(L)$  and  $Z_{n-2}(L)^\perp$ ,  $I$  is an ideal.

We thus have a complete picture of the ideals of  $L$ .

We now focus on the characteristic ideals. It turns out that there are as well always characteristic ideals of dimension 1,  $n$  and  $2n - 1$  when  $2n \geq 10$ .

**Remark.** Notice that if  $I$  is a characteristic ideal then the ideal  $I^\perp$  is also

characteristic. To see this let  $\phi$  be any automorphism of the symplectic alternating algebra  $L$  and let  $a \in I^\perp$ . As  $\phi$  is an automorphism we have that  $\phi(a) \in \phi(I)^\perp = I^\perp$ .

**Theorem 3.2** *Let  $L$  be a nilpotent symplectic alternating algebra of dimension  $2n \geq 10$  that is of maximal class.  $L$  has a chain of characteristic ideals*

$$\{0\} = I_0 < I_1 < \cdots < I_n < I_{n-1}^\perp < \cdots < I_1^\perp < I_0^\perp = L$$

where for  $0 \leq k \leq n$ ,  $I_k$  is isotropic of dimension  $k$ .

**Proof** By Theorems 2.10 and 3.1, we know that we can get such a chain of ideals where all the ideals apart from  $I_1, I_n$  and  $I_{2n-1}$  are characteristic. We want to show that we can choose our chain such that  $I_1, I_n$  and  $I_{2n-1}$  are also characteristic. Let  $x_1, y_1, \dots, x_n, y_n$  be a standard basis such that

$$I_k = Fx_n + Fx_{n-1} + \cdots + Fx_{n+1-k}$$

for  $1 \leq k \leq n$ . Then  $I_4 I_2^\perp = Fx_{n-3}y_{n-2}$  is a characteristic ideal. We claim that this is non-trivial. Otherwise  $x_{n-3}y_{n-2} = 0$  and then  $(x_{n-3}u, y_{n-2}) = 0$  for all  $u \in L$  that implies that  $x_{n-3}L \leq Fx_n + Fx_{n-1}$  and we get the contradiction that  $x_{n-3} \in Z_2(L) = I_3$ . Thus we have got a characteristic ideal of dimension 1, namely  $I_4 I_2^\perp = Z_3(L) \cdot L^2$ . Notice that we are assuming here that  $n \geq 5$ . From this we get that  $(I_4 I_2^\perp)^\perp$  is a characteristic ideal of dimension  $2n - 1$ .

It remains to find a characteristic ideal of dimension  $n$ . We know that  $L^n = Fx_n + Fx_{n-1} + \cdots + Fx_2$ ,  $L^{n-1} = I_{n-1}^\perp = Fx_n + \cdots + Fx_1 + Fy_1$  and  $L^{n-2} = I_{n-2}^\perp = I_{n-1}^\perp + Fy_2$ . As  $L^{n-1} = L^{n-2} \cdot L$  it follows that  $L^n + Fx_1 + Fy_1 = (L^{n-1} + Fy_2)L$  and thus

$$L^n + Fx_1 + Fy_1 = L^n + y_2 L.$$

Thus there exist  $u, v \in L$  such that  $y_2 u + L^n = x_1 + L^n$  and  $y_2 v + L^n = y_1 + L^n$ . Then

$$(y_2 u, x_1) = 0, (y_2 u, y_1) \neq 0, (y_2 v, x_1) \neq 0, (y_2 v, y_1) = 0.$$

Equivalently

$$(x_1 y_2, u) = 0, (x_1 y_2, v) \neq 0, (y_1 y_2, u) \neq 0, (y_1 y_2, v) = 0$$

and this implies that  $x_1y_2, y_1y_2$  are linearly independent (something that will also be useful later). Consider next the 2-dimensional characteristic subspace

$$L^{n-1}L^{n-2} = Fx_1y_2 + Fy_1y_2.$$

Notice that  $L^{n-1}L^{n-2} \leq I_{n-2}$ . Let  $k$  be the smallest positive integer between 1 and  $n-3$  such that  $L^{n-1}L^{n-2} \leq I_{k+1}$ . Let  $J = L^{n-1}L^{n-2} \cap I_k$ . Then  $\dim J = 1$  and there is a unique one-dimensional subspace  $Fu$  of  $Fx_1 + Fy_1$  such that  $FuL^{n-2} = J$ . Now  $I = I_{n-1} + Fu$  is the characteristic ideal of dimension  $n$  that we wanted. Notice that  $I = \{x \in I_{n+1} : xI_{n+2} \subseteq J\}$ .  $\square$

**Remark.** If  $L$  is a nilpotent symplectic alternating algebra of dimension 8 that is of maximal class then there is no characteristic ideal of dimension 1. The reader can convince himself of this by looking at the classification of these algebras given in the last section.

**Corollary 3.3** *Let  $L$  be a nilpotent symplectic alternating algebra of maximal class and dimension  $2n \geq 10$ . The automorphism group of  $L$  is nilpotent-by-abelian.*

**Proof** Consider a chain of characteristic ideals as given in the last theorem

$$\{0\} = I_0 < I_1 < \dots < I_n < I_{n-1}^\perp < I_{n-2}^\perp < \dots < I_0^\perp = L.$$

Consider the ordered basis  $(x_n, x_{n-1}, \dots, x_1, y_1, \dots, y_n)$  associated with this chain, that is  $I_k = Fx_n + Fx_{n-1} + \dots + Fx_{n+1-k}$ . As the ideals in the chain are all characteristic we see that the matrix of any automorphism with respect to that ordered basis will be upper triangular. The result follows.  $\square$

We next move on to presentations of nilpotent symplectic alternating algebras of maximal class. Suppose  $L$  is any nilpotent symplectic alternating algebra with a presentation

$$\mathcal{P} : (x_iy_j, y_k) = \alpha_{ijk}, \quad (y_iy_j, y_k) = \beta_{ijk} \quad 1 \leq i < j < k \leq n.$$

We would like to read from the presentation whether the algebra is of maximal class. This turns out to be possible.

**Theorem 3.4** *Let  $L$  be a nilpotent symplectic alternating algebra of dimension  $2n \geq 8$  given by some nilpotent presentation  $\mathcal{P}$ . The algebra is of maximal class if and only if  $x_{n-2}y_{n-1}, x_{n-3}y_{n-2}, \dots, x_2y_3$  are non-zero and  $x_1y_2, y_1y_2$  are linearly independent.*

**Proof** Let us first see that these conditions are necessary. Suppose that  $L$  is of maximal class. In the proof of Theorem 3.2 we have already seen that  $x_1y_2$  and  $y_1y_2$  must be linearly independent. As before we let  $I_k = Fx_n + \dots + Fx_{n+1-k}$ . As  $x_{n-2} \notin Z(L)$ , we have  $(x_{n-2}y_{n-1}, y_n) \neq 0$  and thus  $x_{n-2}y_{n-1} \neq 0$ . As the terms of the central chain

$$I_0 < I_2 < I_3 < \dots < I_{n-1} < I_{n-1}^\perp < I_{n-2}^\perp < \dots < I_2^\perp < I_0^\perp$$

are the terms of the lower central series, we know that  $I_{k+1}L = I_k$  for  $2 \leq k \leq n-2$ . Thus we have for  $3 \leq k \leq n-2$  that

$$I_{k-1} + Fx_{n-k+1} = (I_k + Fx_{n-k})L.$$

From this it follows  $I_{k-1} + Fx_{n-k+1} = I_{k-1} + x_{n-k}L$ . In particular there exists  $u \in L$  such that  $I_{k-1} + x_{n-k+1} = I_{k-1} + x_{n-k}u$ . It follows that  $0 \neq (x_{n-k}u, y_{n-k+1}) = -(x_{n-k}y_{n-k+1}, u)$ . Hence  $x_{n-k}y_{n-k+1}$  is non-zero for  $3 \leq k \leq n-2$ .

Let us then see that these conditions are sufficient. We do this by showing that  $I_2 = I_3L$ ,  $I_3 = I_4L$ ,  $\dots$ ,  $I_{n-2} = I_{n-1}L$ ,  $I_{n-1} = I_{n-1}^\perp L$ ,  $I_{n-1}^\perp = I_{n-2}^\perp L$ ,  $\dots$ ,  $I_3^\perp = I_2^\perp L$ ,  $I_2^\perp = I_0^\perp L$ . This is sufficient as this would imply that  $L^{2n-3} = I_2 \neq 0$  and thus  $L$  is nilpotent of class  $2n-3$ . Firstly as  $x_{n-2}y_{n-1} \neq 0$  we have that  $(x_{n-2}y_{n-1}, y_n) \neq 0$  and thus  $I_3L = x_{n-2}L = Fx_n + Fx_{n-1} = I_2$ . Now suppose that we have already established that  $I_k = I_{k+1}L$  for all  $2 \leq k \leq m$  where  $2 \leq m \leq n-3$ . Then

$$I_{m+2}L = (I_{m+1} + Fx_{n-m-1})L = I_m + x_{n-m-1}L$$

As  $x_{n-m-1}y_{n-m} \neq 0$  we have  $(x_{n-m-1}u, y_{n-m}) = -(x_{n-m-1}y_{n-m}, u) \neq 0$  for some  $u \in L$  and thus  $I_{m+2}L = I_m + x_{n-m-1}L = I_m + Fx_{n-m} = I_{m+1}$ . We have thus established by induction that

$$I_2 = I_3L, \dots, I_{n-2} = I_{n-1}L.$$

We next show that  $I_{n-1}^\perp L = I_{n-1}$ . As  $x_1y_2 \neq 0$  we have that there exist  $u \in L$  such that  $0 \neq (x_1y_2, u) = -(x_1u, y_2)$  and

$$I_{n-1}^\perp L = (I_{n-1} + Fx_1 + Fy_1)L = I_{n-2} + Fx_2 = I_{n-1}.$$

Next we show that  $I_{n-2}^\perp L = I_{n-1}^\perp$ . As  $x_1 y_2, y_1 y_2$  are linearly independent there exist  $u, v \in L$  such that

$$(x_1 y_2, u) = 0, (x_1 y_2, v) \neq 0, (y_1 y_2, u) \neq 0, (y_1 y_2, v) = 0$$

and thus

$$(y_2 u, x_1) = 0, (y_2 u, y_1) \neq 0, (y_2 v, x_1) \neq 0, (y_2 v, y_1) = 0.$$

Hence

$$I_{n-2}^\perp L = (I_{n-1}^\perp + F y_2) L = I_{n-1} + F y_2 L = I_{n-1} + F x_1 + F y_1 = I_{n-1}^\perp.$$

Now suppose that we have established that  $I_{k-1}^\perp L = I_k^\perp$  for  $m+1 \leq k \leq n-1$  where  $3 \leq m \leq n-2$ . As  $x_{n-m} y_{n-m+1} \neq 0$  it follows that there exists  $u \in L$  such that  $0 \neq (x_{n-m} y_{n-m+1}, u) = (y_{n-m+1} u, x_{n-m})$ . Thus

$$I_{m-1}^\perp L = (I_m^\perp + y_{n-m+1}) L = I_{m+1}^\perp + y_{n-m+1} L = I_{m+1}^\perp + F y_{n-m} = I_m^\perp.$$

It now only remains to see that  $I_0^\perp L = I_2^\perp$ . But this follows from  $x_{n-2} y_{n-1} \neq 0$  that implies that  $(y_{n-1} y_n, x_{n-2}) = (x_{n-2} y_{n-1}, y_n) \neq 0$ . Thus

$$I_0^\perp L = (I_2^\perp + F y_{n-1} + F y_n) L = I_3^\perp + (F y_{n-1} + F y_n) L = I_3^\perp + F y_{n-2} = I_2^\perp.$$

This finishes the proof.  $\square$

**Remark.** In particular it follows that for each  $2n \geq 8$  there exist a nilpotent symplectic alternating algebra of maximal class. One just needs to choose the presentation such that the conditions from Theorem 3.4 hold. One possibility is

$$\begin{aligned} \mathcal{P} : (x_{n-2} y_{n-1}, y_n) &= -1, (x_{n-3} y_{n-2}, y_n) = -1, \dots, (x_1 y_2, y_n) = -1, \\ (y_1 y_2, y_{n-1}) &= -1. \end{aligned}$$

In fact the conditions are not a strong constraint. In particular the values of  $(x_i y_j, y_k), (y_i y_j, y_k)$  where  $j - i \geq 2$  can be chosen freely. The number of such triples is  $2 \binom{n-1}{3}$  that is a polynomial in  $n$  of degree 3 with leading coefficient  $1/3$ . Let  $F$  be any finite field. By a similar argument as we used for determining the growth of nilpotent symplectic alternating algebras we

see that the number  $m(n)$  of nilpotent symplectic alternating algebras of maximal class satisfies

$$m(n) = |F|^{n^3/3 + O(n^2)}.$$

**Remark.** (1) Let  $L$  be a nilpotent symplectic alternating algebra of dimension  $2n \geq 10$  that is of maximal class and consider a chain  $\{0\} = I_0 < \dots < I_n$  of characteristic ideals where  $I_k$  is of dimension  $k$ . We have, for  $4 \leq m \leq n-1$ ,

$$I_m I_{m-2}^\perp = Fx_{n+1-m}y_{n+2-m}$$

and thus we get that  $Fx_2y_3, Fx_3y_4, \dots, Fx_{n-3}y_{n-2}$  are one-dimensional characteristic subspaces of  $L$ . Also

$$I_{n-1}^\perp I_{n-2}^\perp = Fx_1y_2 + Fy_1y_2$$

is a characteristic subspace. So is  $I_n^\perp I_{n-2}^\perp = Fx_1y_2$ .

(2) If  $V$  is a characteristic subspace of dimension  $d$  then we get a chain of characteristic subspaces

$$V \cap I_1 \subseteq V \cap I_2 \subseteq \dots \subseteq V \cap I_n \subseteq V \cap I_{n-1}^\perp \subseteq \dots \subseteq V \cap I_0^\perp = V.$$

Thus there is a chain of characteristic subspaces  $V_1 < V_2 < \dots < V_d$  where  $v_i$  is of dimension  $i$ .

## 4 Nilpotent algebras of dimension $2n \leq 8$

The classification of the nilpotent symplectic alternating algebras of dimensions at most 8 is implicit in [3] although this is not done explicitly and the context there is a more general setting. To demonstrate the machinery that we have developed we will offer a much shorter approach here. The classification of algebras of dimension 10 is far more challenging and will be dealt with in a sequel to this paper. Through this section we will be working with an arbitrary field  $F$ .

We have observed earlier that nilpotent symplectic alternating algebras of dimensions 2 or 4 must be abelian.

## 4.1 Algebras of dimension 6

Let  $L$  be a non-abelian nilpotent symplectic alternating algebra of dimension 6 with a nilpotent presentation  $\mathcal{P}$ . There are at most two non-zero triple values

$$(x_1y_2, y_3) = a, \quad (y_1y_2, y_3) = b.$$

As  $L$  is non-abelian, one of these must be non-zero and, by replacing  $x_1, y_1$  by  $-y_1, x_1$  if necessary, we can assume that  $b \neq 0$ . Replacing then  $x_3, y_3$  by  $bx_3, \frac{1}{b}y_3$  implies that we can further assume that  $(y_1y_2, y_3) = 1$ . Finally replacing  $x_1, y_1$  by  $x_1 - ay_1, y_1$  and we can also assume that  $(x_1y_2, y_3) = 0$ . Apart from the abelian algebra, there is thus only one algebra of dimension 6 with presentation

$$\mathcal{P}_1 : \quad (y_1y_2, y_3) = 1.$$

(We will normally only write down those triples where the value is non-zero).

## 4.2 Algebras of dimension 8

First suppose that  $Z(L)$  is not isotropic. We can then choose our standard basis such that  $I = Fx_4 + Fy_4 \subseteq Z(L)$  and we get a direct sum  $I \oplus I^\perp$  of symplectic alternating algebras of dimensions 2 and 6. From 4.1 we then know that apart from the abelian algebra, there is only one such algebra  $L_2 = Fx_4 + Fx_3 + Fx_2 + Fx_1 + Fy_1 + Fy_2 + Fy_3 + Fy_4$  with presentation

$$\mathcal{P}_2 : \quad (y_1y_2, y_3) = 1.$$

We then turn to the situation where  $Z(L)$  is isotropic. Let us first see that  $\dim Z(L) \neq 4$ . We argue by contradiction and suppose that  $\dim Z(L) = 4$ . Pick a standard basis such that  $Z(L) = Fx_4 + Fx_3 + Fx_2 + Fx_1$ . Now  $L$  is not abelian and thus  $(y_iy_j, y_k) \neq 0$  for some  $1 \leq i < j < k \leq 4$ . Without loss of generality, we can suppose that  $(y_1y_2, y_3) = 1$ . Suppose now that  $(y_1y_2, y_4) = a$ ,  $(y_2y_3, y_4) = b$  and  $(y_3y_1, y_4) = c$ . Let  $\bar{y}_4 = y_4 - by_1 - cy_2 - ay_3$ . Inspection shows that  $\bar{y}_4$  is orthogonal to  $L^2 = Fy_1y_2 + Fy_2y_3 + Fy_3y_1 + F\bar{y}_4y_1 + F\bar{y}_4y_2 + F\bar{y}_4y_3$ . Thus  $\bar{y}_4 \in (L^2)^\perp = Z(L)$  and we get the contradiction that  $\dim Z(L) \geq 5$ . Thus we have shown that  $\dim Z(L) \neq 4$  and as  $\dim Z(L)$  is always at least 2, we have two cases to consider:  $\dim Z(L) = 3$  and  $\dim Z(L) = 2$ .

Dim  $Z(L) = 3$ . We can choose the standard basis such that  $Z(L) = Fx_4 + Fx_3 + Fx_2$  and  $L^2 = Z(L)^\perp = Fx_4 + Fx_3 + Fx_2 + Fx_1 + Fy_1$ . By Theorem 2.10, we know that  $L^3 = L^2L \subseteq Z(L)$  and by Proposition 2.11 we must then have  $L^3 = Z(L)$ . As  $x_1 \notin Z(L)$ , we must have  $(x_1y_i, y_j) \neq 0$  for some  $2 \leq i < j \leq 4$ . Without loss of generality  $(x_1y_2, y_3) \neq 0$ . By replacing  $y_4, y_1$  by  $y_4 - ax_1, y_1 + ax_4$  for a suitable  $a$ , we can assume that  $(y_2y_4, y_3) = 0$ . Let  $V = Fy_2 + Fy_3 + Fy_4$ . Now  $(y_2y_3, y_4) = 0$  and  $L^2 = Z(L) + Fx_1 + Fy_1$ . As  $L = L^2 + V$  it follows that  $V^2 = Fx_1 + Fy_1$  and as  $V^2$  is not isotropic we must have that some two of  $y_2y_3, y_4y_3, y_2y_4$  are not isotropic. Without loss of generality we can suppose that these are  $y_2y_3$  and  $y_4y_3$ . By replacing  $y_4, x_4$  by  $ay_4, \frac{1}{a}x_4$  for a suitable  $a \in F$ , we can furthermore assume that  $(y_2y_3, y_4y_3) = 1$ . Thus

$$Fx_1 + Fy_1 = V^2 = Fy_2y_3 + Fy_4y_3$$

and  $y_2y_4 = ay_2y_3 + by_4y_3$  for some  $a, b \in F$ . It follows that  $(y_2 + by_3)(y_4 - ay_3) = 0$ . Now replace  $y_2, y_4, x_3$  by  $y_2 + by_3, y_4 - ay_3, x_3 - bx_2 + ax_4$  and then replace  $x_1, y_1$  by  $y_2y_3, y_4y_3$ . It follows that we get a new standard basis where

$$y_2y_3 = x_1, \quad y_4y_3 = y_1, \quad y_2y_4 = 0.$$

This implies that the only non-zero triples are  $(y_1y_2, y_3) = 1$  and  $(x_1y_3, y_4) = 1$ . There is thus only one possible candidate here, the algebra  $L_3$  with presentation

$$\mathcal{P}_3 : \quad (y_1y_2, y_3) = 1, \quad (x_1y_3, y_4) = 1.$$

Conversely, one sees by inspection that  $Z(L_3) = Fx_4 + Fx_3 + Fx_2$  and this candidate is a genuine example with  $\dim Z(L) = 3$ .

Dim  $Z(L) = 2$ . We know that the class of  $L$  is at most  $2 \cdot 4 - 3 = 5$  and thus  $L^5 \leq Z(L)$ . Let  $k$  be the smallest positive integer  $2 \leq k \leq 5$  such that  $L^k \leq Z(L)$ . As  $\dim L^k \leq 2$ , it follows from Proposition 2.11 that  $k = 5$ . Hence  $L$  is of maximal class and by Theorem 3.1 we can choose our standard basis such that, we get ideals  $I_k = Fx_n + \dots + Fx_{n+1-k}$ ,  $k = 0, \dots, n$  where

$$\{0\} = I_0 < I_1 < \dots < I_4 = I_4^\perp < I_3^\perp < \dots < I_0^\perp = L$$

is a central series with  $I_2 = Z(L) = L^5$ ,  $I_3 = Z_2(L) = L^4$ ,  $I_3^\perp = Z_3(L) = L^3$  and  $I_2^\perp = Z_4(L) = L^2$ . By Theorem 3.4 we furthermore have that  $x_1y_2, y_1y_2$



are linearly independent and thus a basis for  $Z(L) = Fx_4 + Fx_3$ . We can now pick our standard basis such that  $x_1y_2 = x_4$  and  $y_1y_2 = x_3$ . As  $x_2 \notin Z(L)$ , we also have that  $(x_2y_3, y_4) \neq 0$ . This means that we have the non-zero triples  $(x_1y_2, y_4) = 1$ ,  $(y_1y_2, y_3) = 1$  and  $(x_2y_3, y_4) = r \neq 0$ . The only remaining triples that are possibly non-zero are  $(x_1y_3, y_4)$ ,  $(y_1y_3, y_4)$  and  $(y_2y_3, y_4)$ . Replacing  $x_1, y_1$ , and  $y_2$  by  $x_1 - ax_2$ ,  $y_1 - bx_2$  and  $y_2 + ay_1 - bx_1 - cx_2$  for suitable  $a, b, c \in F$ , we can assume that these extra triples are zero. We can thus choose our basis so that our algebra  $L(r)$  has presentation

$$\mathcal{P}(r) : (x_2y_3, y_4) = r, (x_1y_2, y_4) = 1, (y_1y_2, y_3) = 1.$$

We finally need to sort out when, for  $r, s \in F^* = F \setminus \{0\}$ ,  $L(r)$  and  $L(s)$  are isomorphic. We will see that this happens if and only if  $r/s \in (F^*)^3$ . To see that this is a sufficient condition, suppose we have an algebra  $L$  that has a presentation  $\mathcal{P}(r)$  with respect to some standard basis  $x_1, y_1, x_2, y_2, x_3, y_3, x_4, y_4$ . Suppose that  $s = a^3r$  for some  $a \in F^*$ . Let  $\bar{x}_1 = x_1$ ,  $\bar{y}_1 = y_1$ ,  $\bar{x}_2 = ax_2$ ,  $\bar{y}_2 = \frac{1}{a}y_2$ ,  $\bar{x}_3 = \frac{1}{a}x_3$ ,  $\bar{y}_3 = ay_3$ ,  $\bar{x}_4 = \frac{1}{a}x_4$  and  $\bar{y}_4 = ay_4$ . Inspection shows that  $L$  has presentation  $\mathcal{P}(s)$  with respect to the new basis. Hence  $L(s) \cong L(r)$  when  $r/s \in (F^*)^3$ . It remains to see that the condition is also necessary. Consider the algebra  $L(r)$  and take an arbitrary new standard basis  $\bar{x}_1, \bar{y}_1, \bar{x}_2, \bar{y}_2, \bar{x}_3, \bar{y}_3, \bar{x}_4, \bar{y}_4$  such that  $L(r)$  satisfies the presentation  $\mathcal{P}(s)$  for some  $s \in F^*$ . We want to show that  $s/r \in (F^*)^3$ . Now

$$\bar{y}_3 = ay_3 + by_4 + u, \quad \bar{y}_4 = cy_3 + dy_4 + v$$

for some  $u, v \in L^2$  and  $a, b, c, d \in F$  where  $ad - bc \neq 0$ . As  $\dim L^2 - \dim L^3 = 1$  it follows readily that  $L^2L^2 \leq L^4$  and it follows that

$$\begin{aligned} \bar{y}_3\bar{y}_4\bar{y}_3 &= (ay_3 + by_4)(cy_3 + dy_4)(ay_3 + by_4) + w, \\ \bar{y}_3\bar{y}_4\bar{y}_4 &= (ay_3 + by_4)(cy_3 + dy_4)(cy_3 + dy_4) + z, \end{aligned}$$

for some  $w, z \in L^4$ . As  $L^6 = 0$  we have that  $L^4$  is orthogonal to  $L^3$  and thus in the following direct calculations we can omit  $w$  and  $z$ . We have

$$-s^2 = (\bar{y}_3\bar{y}_4\bar{y}_3, \bar{y}_3\bar{y}_4\bar{y}_4) = -(ad - bc)^3r^2.$$

Hence  $s/r \in (F^*)^3$ .

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